

CONSTRUCTION OF SCHEMOIDS FROM POSETS

NUMATA, YASUhide

ABSTRACT. A schemoid is a generalization of association schemes from the point of view of small categories. In this article, we discuss schemoid structures for two kinds of small categories; the canonical small category defined by a poset, and another small category which arises a poset. We also discuss the schemoid algebra, that is an analogue of the Bose–Mesner algebra for an association scheme, for them.

1. INTRODUCTION

We call a pair of a finite set X and a partition $S = \{R_0, R_1, \dots, R_n\}$ of $X \times X$ an *n-class association scheme* if S satisfies the following:

- (1) $R_0 = \{(x, x) \mid x \in X\}$.
- (2) If $R \in S$, then $\{(y, x) \mid (x, y) \in R\} \in S$.
- (3) For $i, j, k \in \{0, 1, \dots, n\}$, there exists $p_{i,j}^k$ such that

$$p_{i,j}^k = \#\{(x, y), (y, z) \in R_i \times R_j\}$$

for every $(x, z) \in R_k$.

Association schemes are introduced by Bose and Shimamoto [3] in their study of design of experiments. Since we can regard association schemes as a generalization of combinatorial designs, groups, and so on (see also [1]), many authors study association schemes from the view point of algebraic combinatorics. Bose and Mesner introduced an algebra which arises an association scheme [2]. The algebra is called the Bose–Mesner algebra, and plays an important role for algebraic study for association schemes. In [5], Kuribayashi and Matsuo introduced an association schemoid and a quasi-schemoid, which are generalizations of an association scheme from the viewpoint of small categories. A quasi-schemoid, we call it a schemoid for short in this paper, is defined to be a pair of small category \mathcal{C} and a partition S of the morphisms of \mathcal{C} satisfying the following condition:

- $\{(f, g) \in \sigma \times \tau \mid f \circ g = h\}$ and $\{(f, g) \in \sigma \times \tau \mid f \circ g = k\}$ have the same cardinality for $\sigma, \tau, \mu \in S$ and $h, k \in \mu$.

Let $(X, \{R_0, R_1, \dots, R_n\})$ be an n -class association scheme. Consider the codiscrete groupoid \mathcal{C}_X on X , i.e., the small category such that $\text{Obj}(\mathcal{C}_X) = X$ and $\text{Hom}_{\mathcal{C}_X}(x, y) = \{(y, x)\}$. For $R_i \in S$, we define σ_i to be the set of morphisms (x, y) in \mathcal{C}_X such that $(x, y) \in R_i$. Since the compositions of morphisms in \mathcal{C}_X are defined by $(x, y) \circ (y, z) = (x, z)$, the pair $(\mathcal{C}_X, \{\sigma_0, \sigma_1, \dots, \sigma_n\})$ is a schemoid. Hence we can identify an association scheme with a schemoid. Moreover we can also construct a schemoid from a coherent configuration in the same manner. A schemoid requires the condition which is analogue of Condition 3 in the definition of an association scheme. An association schemoid is a schemoid satisfying the other conditions in the definition of an association scheme. Kuribayashi and Matsuo also introduce a subalgebra of the category algebra which arises a schemoid. The algebra

2010 *Mathematics Subject Classification.* 18A99, 18A32, 18D99.

Key words and phrases. Association schemes; Bose–Mesner algebras; Quasi-schemoids.

This work was supported by JSPS KAKENHI Grant Number 25800009.

is an analogue of the Bose–Mesner algebra for an association scheme. Kuribayashi [4] and Kuribayashi–Momose [6] develop homotopy theory for schemoids.

The purpose of this article is to give examples of schemoids. In this paper, we discuss schemoid structures for two kinds of small categories; the canonical small category obtained from a poset, and another acyclic small category obtained from a poset. The organization of this article is the following: We define schemoids and schemoid algebras in Section 2. In Section 3, we discuss schemoid structures on two kinds of small categories which arises a poset.

Acknowledgments. The author thanks anonymous referees for their helpful comments.

2. DEFINITION

Here we recall the definition of small categories, schemoids and schemoid algebras.

First we recall small categories and functors. A *small category* \mathcal{C} is a quintuple of the set $\text{Obj}(\mathcal{C})$ of objects, the set $\text{Mor}(\mathcal{C})$ of morphisms, maps $s: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ and $t: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$, and the operation \circ of composition, satisfying the following properties: For each morphism $f \in \text{Mor}(\mathcal{C})$, $s(f)$ is called the *source* of f , and $t(f)$ is called the *target* of f . A morphism f is called an *endomorphism* if $s(f) = t(f)$. For $x, y \in \text{Obj}(\mathcal{C})$, define $\text{Hom}_{\mathcal{C}}(x, y) = \{ f \in \text{Mor}(\mathcal{C}) \mid s(f) = x, t(f) = y \}$. A sequence $(f_n, f_{n-1}, \dots, f_1)$ of morphisms is called a *nerve* if $t(f_i) = s(f_{i+1})$ for $i = 1, \dots, n-1$. The composition $g \circ f$ is defined for each nerve (g, f) of length 2. The composition $g \circ f$ is in $\text{Hom}_{\mathcal{C}}(x, z)$ for $g \in \text{Hom}_{\mathcal{C}}(y, z)$ and $f \in \text{Hom}_{\mathcal{C}}(x, y)$. Moreover the operation satisfies $(h \circ g) \circ f = h \circ (g \circ f)$ for every nerve (h, g, f) of length 3. For $x \in \text{Obj}(\mathcal{C})$, a morphism from x to x is called an *endomorphism* on x . For each $x \in \text{Obj}(\mathcal{C})$, there uniquely exists an endomorphism id_x on x such that $\text{id}_x \circ f = f$ for every f with $t(f) = x$ and $g \circ \text{id}_x = g$ for every g with $s(g) = x$. The morphism id_x is called the *identity* on x .

Let \mathcal{C} and \mathcal{C}' be small categories. We call a pair $\varphi = (\varphi^{\text{Obj}}, \varphi^{\text{Mor}})$ a *functor* from \mathcal{C} to \mathcal{C}' if the map φ^{Obj} from $\text{Obj}(\mathcal{C})$ to $\text{Obj}(\mathcal{C}')$ and the map φ^{Mor} from $\text{Mor}(\mathcal{C})$ to $\text{Mor}(\mathcal{C}')$ satisfy the following:

- (1) $\varphi^{\text{Mor}}(\text{id}_x) = \text{id}_{\varphi^{\text{Obj}}(x)}$ for each $x \in \text{Obj}(\mathcal{C})$.
- (2) $\varphi^{\text{Mor}}(f \circ g) = \varphi^{\text{Mor}}(f) \circ \varphi^{\text{Mor}}(g)$ for all nerve (f, g) of length 2 in \mathcal{C} .

Next we define schemoids. In this paper, we define a schemoid as the pair of a small category \mathcal{C} and a map π from the set $\text{Mor}(\mathcal{C})$ of morphisms to a set I . For a map π , we obtain a partition $\{ \pi^{-1}(\{i\}) \mid i \in I \}$ of $\text{Mor}(\mathcal{C})$. On the other hand, for a partition S of $\text{Mor}(\mathcal{C})$, we obtain the canonical surjection from $\text{Mor}(\mathcal{C})$ to S . Via this translation, the following definition is equivalent to the original definition of a schemoid.

Definition 2.1. Let \mathcal{C} be a small category, I a set, and π a map from the set $\text{Mor}(\mathcal{C})$ of morphisms in \mathcal{C} to the set I . For $i, j \in I$ and $h \in \text{Mor}(\mathcal{C})$, we define $N_h^{i,j}$ to be

$$\left\{ (f, g) \in \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) \left| \begin{array}{l} \pi(f) = i, \\ \pi(g) = j, \\ f \circ g = h. \end{array} \right. \right\}.$$

We call the triple (\mathcal{C}, I, π) a *schemoid* if

$$\pi(h) = \pi(k) \implies N_h^{i,j} \text{ and } N_k^{i,j} \text{ have the same cardinality}$$

for each $i, j \in I$ and $h, k \in \text{Mor}(\mathcal{C})$.

For a morphism f of a small category \mathcal{C} , we write \mathcal{C}_f to denote the minimum subcategory of \mathcal{C} such that $\text{Mor}(\mathcal{C}_f)$ contains

$$\{ g \in \text{Mor}(\mathcal{C}) \mid f_1 \circ g \circ f_2 = f \text{ for some } f_1, f_2 \in \text{Mor}(\mathcal{C}) \}.$$

Then we can show the following lemma.

Lemma 2.2. *Let \mathcal{C} be a small category which does not contain any endomorphism except identities. Let π be a map from the set $\text{Mor}(\mathcal{C})$ of morphisms to a set I . If the following condition holds, then (\mathcal{C}, I, π) is a schemoid: For all morphisms f and g such that $\pi(f) = \pi(g)$, there exists a functor $\varphi_{f,g}$ from \mathcal{C}_f to \mathcal{C}_g such that*

- (1) $\varphi_{f,g}^{\text{Mor}}$ is a bijection;
- (2) $\pi(f') = \pi(\varphi_{f,g}^{\text{Mor}}(f'))$ for each morphism f' in \mathcal{C}_f ; and
- (3) $\varphi_{f,g}^{\text{Mor}}(f) = g$.

Proof. Let $h, k \in \text{Mor}(\mathcal{C})$ satisfy $\pi(h) = \pi(k)$. The map $\varphi_{h,k}^{\text{Mor}}$ induces a bijection from $N_h^{i,j}$ to $N_k^{i,j}$ for each $i, j \in I$. Hence the triple (\mathcal{C}, I, π) is a schemoid. \square

For a small category \mathcal{C} and a field \mathbb{K} , define $\mathbb{K}[\mathcal{C}]$ to be the \mathbb{K} -vector space whose basis is $\text{Mor}(\mathcal{C})$. We define the product by

$$g \cdot f = \begin{cases} g \circ f & s(g) = t(f) \\ 0 & s(g) \neq t(f) \end{cases}$$

for $f, g \in \text{Mor}(\mathcal{C})$. Moreover, for $\sum_{f \in \text{Mor}(\mathcal{C})} \alpha_f f$ and $\sum_{g \in \text{Mor}(\mathcal{C})} \beta_g g \in \mathbb{K}[\mathcal{C}]$, we define the product of them by

$$\left(\sum_{f \in \text{Mor}(\mathcal{C})} \alpha_f f \right) \cdot \left(\sum_{g \in \text{Mor}(\mathcal{C})} \beta_g g \right) = \sum_{f \in \text{Mor}(\mathcal{C})} \sum_{g \in \text{Mor}(\mathcal{C})} (\alpha_f \beta_g) f \cdot g.$$

If $\text{Obj}(\mathcal{C})$ is a finite set, then $\mathbb{K}[\mathcal{C}]$ is a \mathbb{K} -algebra with the unit $\sum_{x \in \text{Obj}(\mathcal{C})} \text{id}_x$. Let π be a map from $\text{Mor}(\mathcal{C})$ to a set I . Assume that $\pi^{-1}(\{i\})$ is finite for every $i \in I$. For $i \in I$, we define \tilde{i} to be $\sum_{f: \pi(f)=i} f \in \mathbb{K}[\mathcal{C}]$. We define $\mathbb{K}(\mathcal{C}, I, \pi)$ to be the vector subspace of $\mathbb{K}[\mathcal{C}]$ spanned by $\{\tilde{i} \mid i \in I\}$. For a schemoid (\mathcal{C}, I, π) such that $\pi^{-1}(\{i\})$ is finite for every $i \in I$, $\mathbb{K}(\mathcal{C}, I, \pi)$ is a subalgebra of $\mathbb{K}[\mathcal{C}]$. (The subalgebra $\mathbb{K}(\mathcal{C}, I, \pi)$ may not have the unit.) We call $\mathbb{K}(\mathcal{C}, I, \pi)$ a *schemoid algebra*.

3. SCHEMOIDS CONSTRUCTED FROM POSETS

Here we consider two kinds of small categories defined from a poset. The prototypical example of them is a schemoid structure for the n -th Boolean lattice $2^{[n]}$, i.e., the poset consisting of all subsets of $\{1, \dots, n\}$ ordered by inclusion. For $X, Y \in 2^{[n]}$, we can consider the set difference $X \setminus Y$. In 3.1, we discuss a poset with the operation which is analogue of the operation of set difference. The operation induces a schemoid structure for the canonical small category obtained from the poset. On the other hand, for $X, Y \in 2^{[n]}$ with $X \cap Y = \emptyset$, a greater element $X \cup Y$ than X is obtained from X by adding Y . By an analogue of the operation, we introduce an acyclic small category obtained from a poset with some conditions in 3.2. The category has also a schemoid structure.

3.1. Posets as a small category. Let P be a poset with respect to \leq . For $x, y \in P$, we define the interval $[x, y]$ from x to y by $[x, y] = \{z \mid x \leq z \leq y\}$. We can naturally regard the poset P as the following small category \mathcal{C}_P : the set $\text{Obj}(\mathcal{C}_P)$ of objects is P and the set $\text{Mor}(\mathcal{C}_P)$ of morphisms is the relation \geq , i.e., $\{(y, x) \mid x \leq y\} \subset P \times P$. For $x \leq y \in P$, $\text{Hom}_{\mathcal{C}_P}(x, y)$ consists of (y, x) . For $(y, x) \in \text{Hom}_{\mathcal{C}_P}(x, y)$ and $(z, y) \in \text{Hom}_{\mathcal{C}_P}(y, z)$, it follows by definition that $x \leq z$.

We define the composition $(z, y) \circ (y, x)$ by $(z, y) \circ (y, x) = (z, x)$. For $x \in P$, id_x is (x, x) .

Here we consider a poset P with a difference operation δ defined as follows:

Definition 3.1. Let o be an element in the poset P , and δ a map from the set $\{(y, x) \in P^2 \mid x \leq y\}$ to P . We say that δ is a *difference operation with the base point o* if there exists a family

$$\{\varphi_{x,y}: [x, y] \rightarrow [o, \delta(y, x)] \mid x \leq y\}$$

of maps satisfying the following:

- (1) Each $\varphi_{x,y}$ is a bijection from the interval $[x, y]$ to the interval $[o, \delta(y, x)]$.
- (2) $\delta(o, \varphi_{x,y}(z)) = \delta(x, z)$ for $x \leq z \leq y$.

Let δ be a difference operation of poset P . In this case, we have bijections $\varphi_{x,y}$. If we fix an interval $[x, y]$, then we can translate each element in the interval $[x, y]$ into some interval from the base point o via the bijection $\varphi_{x,y}$. Fix $x \in P$ and consider two intervals $[x, y]$ and $[x, y']$. For $z \in [x, y] \cap [x, y']$, it follows by Condition 2 that $\delta(o, \varphi_{x,y}(z)) = \delta(o, \varphi_{x,y'}(z))$. In this sense, Condition 2 implies that the translation depends not on the interval but only on the minimum of the interval.

Since the difference operation induces functors $\varphi_{f,g}$ from $(\mathcal{C}_P)_f$ to $(\mathcal{C}_P)_g$, Theorem 3.2 follows from Lemma 2.2.

Theorem 3.2. For a poset P with the difference operation δ , the triple $(\mathcal{C}_P, P, \delta)$ is a schemoid.

Example 3.3. Let P be the n -th Boolean lattice, i.e., $2^{[n]}$ ordered by inclusion. For $x \leq y \in P$, we define $\delta(y, x)$ to be $y \setminus x$. The map δ is a difference operation with the base point \emptyset . Hence $(\mathcal{C}_P, P, \delta)$ is a schemoid. In this case, the schemoid algebra $\mathbb{K}(\mathcal{C}_P, P, \delta)$ is isomorphic to $\mathbb{K}[x_i \mid i \in P] / (x_i^2 \mid i \in P)$.

Example 3.4. Let P be a Coxeter groups ordered by the Bruhat order. For $x \leq y \in P$, we define $\delta(y, x)$ to be yx^{-1} . The map δ is a difference operation with the base point ε . Hence $(\mathcal{C}_P, P, \delta)$ is a schemoid. In this case, the schemoid algebra $\mathbb{K}(\mathcal{C}_P, P, \delta)$ is isomorphic to the NilCoxeter algebra.

Example 3.5. Let Δ be a simplicial complex on the vertex set V . Consider the lattice P of faces of the simplicial complex Δ . (We regard \emptyset as a face of Δ .) For $x, y \in P$, we define $\delta(y, x)$ to be $y \setminus x$. The map δ is a difference operation with the base point $o = \emptyset$. Hence $(\mathcal{C}_P, P, \delta)$ is a schemoid. Let I_Δ be an ideal of $\mathbb{K}[x_i \mid i \in V]$ generated by $\{x_{v_1} \cdots x_{v_l} \mid \{v_1, \dots, v_l\} \notin \Delta\}$. The quotient ring $\mathbb{K}[x_i \mid i \in V] / I_\Delta$ is called the *Stanley–Reisner ring*. Let $\tilde{I}_\Delta = I_\Delta + (x_i^2 \mid i \in V)$. The schemoid algebra $\mathbb{K}(\mathcal{C}_P, P, \delta)$ is isomorphic to $\mathbb{K}[x_i \mid i \in V] / \tilde{I}_\Delta$.

Remark 3.6. In Appendix of [6], Kuribayashi and Momose discuss schemoids in Example 3.5 from the point of view of the category theory.

3.2. Yet another small category obtained from posets. Here we introduce another kind of small categories obtained from a poset. We also introduce a schemoid structure for it.

Let P be a poset with respect to \leq . Assume that the number of minimal elements in $\{z \in P \mid x \leq z, y \leq z\}$ is 1 or 0 for each pair $x, y \in P$. We write $x \vee y$ to denote the minimum element in $\{z \in P \mid x \leq z, y \leq z\}$ if $\{z \in P \mid x \leq z, y \leq z\} \neq \emptyset$. Assume the following conditions:

- (1) P has the minimum element $\hat{0}$.
- (2) P is a ranked poset with the rank function ρ .
- (3) $\rho(x \vee y) \leq \rho(x) + \rho(y)$ for $x, y \in P$.

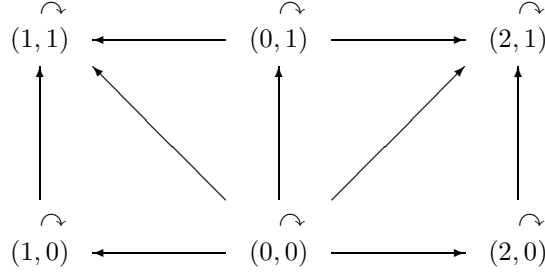


FIGURE 1. The small category in Example 3.7

We define a small category \tilde{P} whose set of objects is P . For $x, y, d \in P$ such that $\rho(y) = \rho(x) + \rho(d)$ and $y = x \vee d$, we define a morphism $f_{x,y}^d$ from x to y . Or equivalently,

$$\text{Hom}_{\tilde{P}}(x, y) = \left\{ f_{x,y}^d \mid \begin{array}{l} d \in P. \\ y = x \vee d. \\ \rho(y) = \rho(x) + \rho(d). \end{array} \right\}$$

for $x, y \in P$. If $f_{x,y}^c$ and $f_{y,z}^d \in \text{Mor}(\tilde{P})$, then $\rho(y) = \rho(x) + \rho(c)$ and $\rho(z) = \rho(y) + \rho(d)$. Hence $\rho(z) = \rho(x) + \rho(c) + \rho(d)$. Since $z = x \vee (c \vee d)$, we have $\rho(x) + \rho(c) + \rho(d) = \rho(z) \leq \rho(x) + \rho(c \vee d)$. On the other hand, $\rho(x) + \rho(c \vee d) \leq \rho(x) + \rho(c) + \rho(d)$ since $\rho(c \vee d) \leq \rho(c) + \rho(d)$. Hence $\rho(z) = \rho(x) + \rho(c \vee d) = \rho(x) + \rho(c) + \rho(d)$. Since $f_{x,z}^{d \vee c}$ is in $\text{Mor}(\tilde{P})$, we define the composition $f_{y,z}^d \circ f_{x,y}^c$ to be $f_{x,z}^{d \vee c}$.

Example 3.7. Let $P = \{0, 1, 2\} \times \{0, 1\}$. For $(x, y), (x', y') \in P$, $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. In this case, the set of morphisms of \tilde{P} consists of the following:

$$\begin{aligned} &f_{(0,0),(0,1)}^{(0,1)}, f_{(1,0),(1,1)}^{(0,1)}, f_{(2,0),(2,1)}^{(0,1)}, \\ &f_{(0,0),(1,0)}^{(1,0)}, f_{(0,1),(1,1)}^{(1,0)}, \\ &f_{(0,0),(2,0)}^{(2,0)}, f_{(0,1),(2,1)}^{(2,0)}, \\ &f_{(0,0),(1,1)}^{(1,1)}, \\ &f_{(0,0),(2,1)}^{(2,1)}, \end{aligned}$$

and identities. See also Figure 1.

Theorem 3.8. For the map π from $\text{Mor}(\tilde{P})$ to P defined by $\pi(f_{x,y}^d) = d$, the triple (\tilde{P}, P, π) is a schemoid.

Proof. Let $f_{x,x \vee d}^d$ and $f_{x',x' \vee d}^d$ be in $\text{Mor}(\tilde{P})$. In this case, it follows from the definition of morphisms in \tilde{P} that

$$\begin{aligned} \rho(x \vee d) &= \rho(x) + \rho(d), \\ \rho(x' \vee d) &= \rho(x') + \rho(d). \end{aligned}$$

Let $f_{x,x \vee d}^d = f_{x \vee d_1, x \vee d_1 \vee d_2}^{d_2} \circ f_{x, x \vee d_1}^{d_1}$. In this case, we have $d = d_1 \vee d_2$ and $\rho(d) = \rho(d_1) + \rho(d_2)$. If $\rho(x' \vee d_1) < \rho(x') + \rho(d_1)$ or $\rho(x' \vee d_1 \vee d_2) < \rho(x' \vee d_1) + \rho(d_2)$, then we have

$$\rho(x' \vee d_1 \vee d_2) < \rho(x') + \rho(d_1) + \rho(d_2) = \rho(x') + \rho(d_1 \vee d_2),$$

which contradicts $\rho(x' \vee d) = \rho(x') + \rho(d)$. Hence morphisms $f_{x' \vee d_1, x' \vee d_1 \vee d_2}^{d_2}$ and $f_{x', x' \vee d_1}^{d_1}$ satisfies $f_{x', x' \vee d}^d = f_{x' \vee d_1, x' \vee d_1 \vee d_2}^{d_2} \circ f_{x', x' \vee d_1}^{d_1}$. Therefore there exists a bijection between $N_{f_{x, x \vee d}^d}^{d_1, d_2}$ and $N_{f_{x', x' \vee d}^d}^{d_1, d_2}$. Hence the triple (\tilde{P}, P, π) is a schemoid. \square

Now we discuss the schemoid algebra. Consider the polynomial ring $\mathbb{K}[X_x | x \in P]$ in variables corresponding to elements in P . Define G_i by

$$\begin{aligned} G_0 &= \{ X_0 - 1 \} \\ G_1 &= \{ X_x X_y \mid \rho(x \vee y) < \rho(x) + \rho(y) \} \\ G'_1 &= \{ X_x X_y \mid \{ z \mid z \geq x, z \geq y \} = \emptyset \} \\ G_2 &= \{ X_x X_y - X_{x \vee y} \mid \rho(x \vee y) = \rho(x) + \rho(y) \}. \end{aligned}$$

Let I be the ideal generated by $G_0 \cup G_1 \cup G'_1 \cup G_2$, and R_P the quotient ring $\mathbb{K}[X_x | x \in P]/I$. The ring R_P is the same as the ring defined in the following manner: R_P is the \mathbb{K} -vector space whose basis is $\{ X_x \mid x \in P \}$

$$\begin{cases} X_x X_y - X_{x \vee y} & (\text{if there exists } x \vee y \text{ and } \rho(x \vee y) = \rho(x) + \rho(y)), \\ X_x X_y = 0 & (\text{otherwise}). \end{cases}$$

Theorem 3.9. *For the map π from $\text{Mor}(\tilde{P})$ to P defined by $\pi(f_{x,y}^d) = d$, the schemoid algebra for the schemoid (\tilde{P}, P, π) is isomorphic to R_P .*

Example 3.10. Let P be the n -th Boolean lattice $2^{[n]}$. In this case, the set of morphisms is

$$\{ f_{x, x \cup d}^d \mid x, d \subset [n], x \cap d = \emptyset \}.$$

Hence (\tilde{P}, P, π) is (\mathcal{C}_P, P, π) in Example 3.4.

Example 3.11. Let K be a finite field. Consider the poset P of all subspaces in K^n ordered by the inclusion. In this case, the set of morphisms is

$$\{ f_{V, V \oplus W}^W \mid V, W \in P, V \cap W = 0 \}.$$

Example 3.12. Let P be the poset of flats of a matroid M ordered by inclusion. Assume that P satisfies the conditions in this section. In this case, the schemoid algebra for (\tilde{P}, P, π) is isomorphic to the algebra defined in Maeno–Numata [7], which is Möbius algebra with the relations $x_i^2 = 0$ for all variables x_i .

REFERENCES

- [1] E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes, Benjamin–Cummings Lecture Note Ser. 58, London, Benjamin, 1984.
- [2] R. C. Bose and Dale M. Mesner. On linear associative algebras corresponding to association schemes of partially balanced designs. *Ann. Math. Statist.*, 30:21–38, 1959.
- [3] R. C. Bose and T. Shimamoto. Classification and analysis of partially balanced incomplete block designs with two associate classes. *J. Amer. Statist. Assoc.*, 47:151–184, 1952.
- [4] K. Kuribayashi, On strong homotopy for quasi-schemoids. *Theory Appl. Categ.* 30 (2015), 114.
- [5] K. Kuribayashi and K. Matsuo. Association schemoids and their categories. *Appl. Categ. Structures* 23 (2015), no. 2, 107136
- [6] K. Kuribayashi and Y. Momose. On Mitchell’s embedding theorem for a quasi-schemoid. Preprint. [arXiv:1507.01745v2](https://arxiv.org/abs/1507.01745v2).
- [7] T. Maeno and Y. Numata. Sperner property and finite-dimensional Gorenstein algebras associated to matroids. Preprint to appear in Journal of Commutative Algebra. [arXiv:1107.5094](https://arxiv.org/abs/1107.5094)

(Numata) DEPARTMENT OF MATHEMATICAL SCIENCES, SHINSHU UNIVERSITY, 3-1-1 ASAHI, MATSUMOTO-SHI, NAGANO-KEN, 390-8621, JAPAN.

E-mail address: nu@math.shinshu-u.ac.jp